B.Sc. Part III (Hons.) $5^{th}$ Paper

## Infinite Products (contd.)

**Q.** Prove that the product $\prod_{1}^{\infty} \left(1 - \frac{x}{c+n}\right) e^{\frac{x}{n}}$ is absolutely convergent where c is a constant other than a (-)ve integer.

**Soln.** Let $\prod (1+U_n) = \prod \left(1 - \frac{x}{c+n}\right) e^{\frac{x}{n}}$

$$\Rightarrow 1+U_n = \left(1 - \frac{x}{c+n}\right) e^{\frac{x}{n}}$$

$$= \left(1 - \frac{x}{c+n}\right)\left(1 + \frac{x}{n} + \frac{x^2}{\lfloor 2\, n^2} + \cdots \text{to } \infty\right)$$

$$= \left(1 + \frac{x}{n} + \frac{x^2}{2n^2} + \cdots \text{to } \infty\right) - \left(\frac{x}{c+n} + \frac{x^2}{n(c+n)} + \frac{x^3}{2n^2(c+n)} + \cdots \text{to } \infty\right)$$

$$\Rightarrow 1+U_n = 1 + x\left(\frac{1}{n} - \frac{1}{c+n}\right) + x^2\left[\frac{1}{2n^2} - \frac{1}{n(c+n)}\right] + \cdots \text{to } \infty$$

$$\Rightarrow 1+U_n = 1 + x\left[\frac{c+n-n}{n(c+n)}\right] + x^2\left[\frac{(c+n) - 2n}{2n^2(c+n)}\right] + \cdots$$

$$\Rightarrow 1+U_n = 1 + x \cdot \frac{c}{n(c+n)} + \frac{x^2(c-n)}{2n^2(c+n)} + \cdots \text{to } \infty$$

$$\Rightarrow U_n = x \cdot \frac{c}{n(c+n)} + x^2 \cdot \frac{c-n}{2n^2(c+n)} + \cdots \text{to } \infty$$

Let $\sum V_n = \sum \frac{1}{n^2}$ be another series

$\therefore V_n = \frac{1}{n^2}$

$$\therefore \frac{|U_n|}{V_n} = \left| \frac{cx \cdot n^2}{n(c+n)} + x^2 \cdot \frac{(c-n)\,n^2}{2n^2(c+n)} + \cdots \text{ to } \infty \right|$$

$$\Rightarrow \frac{|U_n|}{V_n} = \left| \frac{cxn}{c+n} + x^2 \cdot \frac{c-n}{2(c+n)} + \cdots \text{ to } \infty \right|$$

$$\Rightarrow \frac{|U_n|}{V_n} = \left| \frac{cx}{1+\frac{c}{n}} + \frac{x^2}{2} \cdot \frac{\frac{c}{n}-1}{\frac{c}{n}+1} + \cdots \text{ to } \infty \right|$$

$$\Rightarrow \lim_{n\to\infty} \frac{|U_n|}{V_n} = cx + \frac{x^2}{2} \cdot (-1) = cx - \frac{x^2}{2}$$

Case I Let $c$ be a constant.

$\Rightarrow \lim_{n\to\infty} \frac{|U_n|}{V_n} = cx - \frac{x^2}{2}$ which is finite and non-zero for all finite values of $x$.

$\Rightarrow \sum |U_n|$ and $\sum V_n$ behave alike.

But $\sum V_n = \sum \frac{1}{n^2}$ is cgt

$\Rightarrow \sum |U_n|$ is convergent

$\Rightarrow \sum U_n$ is abs. convergent

Case II

Let $c = 0$

then $1 + u_n = \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$

$$\Rightarrow 1 + u_n = \left(1 - \frac{x}{n}\right)\left(1 + \frac{x}{n} + \frac{x^2}{\lfloor 2\, n^2} + \cdots \text{to } \infty\right)$$

$$= \left(1 + \cancel{\frac{x}{n}} + \frac{x^2}{2n^2} + \cdots\right) - \left(\cancel{\frac{x}{n}} + \frac{x^2}{n^2} + \frac{x^3}{2n^3} + \cdots\right)$$

$$\Rightarrow 1 + u_n = 1 - \frac{x^2}{2n^2} + \cdots$$

$$\Rightarrow u_n = -\frac{x^2}{2n^2} + \cdots$$

Let $v_n = \frac{1}{n^2}$

$\Rightarrow \frac{|u_n|}{v_n} = +\left|\frac{x^2}{2}\right| +$ terms containing $n$ in the denominator

$\Rightarrow \lim_{n \to \infty} \frac{|u_n|}{v_n} = +\left|\frac{x^2}{2}\right| = \frac{x^2}{2}$ which is finite and non-zero for all finite values of $x$.

$\Rightarrow \sum |u_n|$ is cgt as $\sum v_n = \sum \frac{1}{n^2}$ is cgt

$\Rightarrow \prod (1 + u_n)$ converges absolutely.